"*On Curves over Finite Fields*"

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Abstract

Let $\mathbb{F}_\ell$ denote the finite field with ℓ elements. For a projective curve C over $\mathbb{F}_\ell$, nonsingular and geometrically irreducible, we have the famous Hasse-Weil upper bound on the number $\#C(\mathbb{F}_\ell)$ of $\mathbb{F}_\ell$-rational points of the curve:

$$\#C(\mathbb{F}_\ell) \leq 1 + \ell + 2\sqrt{\ell} \cdot g(C), \tag{1}$$

where $g(C)$ denotes the genus.

In case ℓ is a square, say $\ell = q^2$, the upper bound (1) is attained; consider the projective curve $\mathcal{H}$ over $\mathbb{F}_\ell$ given by the equation:

$$f(X,Y) = Y^q + Y - X^{q+1}. \tag{2}$$

This is the so-called Hermitian curve $\mathcal{H}$ over $\mathbb{F}_{q^2}$. It satisfies:

$$g(\mathcal{H}) = \frac{q(q-1)}{2} \quad \text{and} \quad \#\mathcal{H}(\mathbb{F}_{q^2}) = 1 + q^3.$$

A tower of curves $\mathcal{F}$ over a finite field $\mathbb{F}_\ell$ is an infinite sequence

$$\mathcal{F} = (\cdots \to C_{n+1} \to C_n \to \cdots \to C_3 \to C_2 \to C_1$$

of projective curves C_n, nonsingular and geometrically irreducible, and surjective maps $\varphi_n \colon C_{n+1} \to C_n$, both defined over $\mathbb{F}_\ell$, such that $g(C_n) \to \infty$ as $n \to \infty$.

The fundamental invariant of a tower $\mathcal{F}$ over $\mathbb{F}_\ell$ is its limit $\lambda(\mathcal{F})$:

$$\lambda(\mathcal{F}) = \lim_{n\to\infty} \frac{\#C_n(\mathbb{F}_\ell)}{g(C_n)}.$$

We have the following Drinfeld-Vladut upper bound:

$$\lambda(\mathcal{F}) \leq \sqrt{\ell} - 1, \quad \text{for any tower } \mathcal{F} \text{ over } \mathbb{F}_\ell. \tag{3}$$

In case ℓ is a square, say $\ell = q^2$, the bound (3) is attained; i.e., we have:

$$\lambda(\mathcal{F}) = q - 1, \quad \text{for some towers } \mathcal{F} \text{ over } \mathbb{F}_{q^2}. \tag{4}$$

Equality (4) was firstly shown by Ihara using modular curves to produce good towers over $\mathbb{F}_{q^2}$ and later by Garcia-Stichtenoth using explicit equations for the curves C_n in the tower. These explicit equations are all related to Equation (2) for the Hermitian curve $\mathcal{H}$ over $\mathbb{F}_{q^2}$.

Much less was known when ℓ is a nonsquare. Recently in a joint work with Bassa, Beelen and Stichtenoth, we have constructed a tower $\mathcal{F}$ over $\mathbb{F}_\ell$ where

$$\ell = p^{2m+1} \quad \text{with } p \text{ prime and } m \geq 1,$$

whose limit satisfies:

$$\lambda(\mathcal{F}) \geq \frac{2(p^{m+1}-1)}{p+1+\varepsilon} \quad \text{where} \quad \varepsilon = \frac{p-1}{p^m-1}. \tag{5}$$

Equation (5) represents a big improvement on former results on towers over nonprime finite fields. The tower $\mathcal{F}$ giving (5) is constructed with explicit equations for the projective curves C_n and it is also inspired by the tower of Garcia-Stichtenoth giving Equality (4).